Adaptive Mesh De-Resolution

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Abstract

The finite element method (FEM) is a triumph of modern engineering that allows us to computationally solve a plurality of partial differential equations (PDE) and physical system models. It would not be an overstatement to say this technique alone enables and supports the entire infrastructure of modern society. Despite its flexibility and ubiquity, the method comes at high computational cost: it seemingly requires denser meshes to decrease numerical error. In this paper we demonstrate that contrary to established hearsay, sparser meshes are actually what is needed to decrease numerical error. We introduce the adaptive mesh de-resolution technique which provably decreases all numerical error to exactly zero.

1. Introduction

The finite element method [**?**] is used ubiquitously as *the* technique to solve partial differential equations (PDE) with no analytic solution. These equations and their solutions are used to model nonlinear dynamics such as (but not limited to) fluid flow [**?**], turbulence [**?**], small and large deformation elastics [**?**], bending instabilities [**?**], acoustics [**?**], and thermo-visco-elastic deformation [**?**]. A variety of softwares have been written to implement the many variations of the finite element method e.g. PolyFEM [**?**], etc. As such they are essential to the function of modern engineering.

Despite the many benefits of FEM, this technique suffers from one critical weakness: it's solution converges to the continuous solution of a PDE in the limit as the underlying mesh density is increased indefinitely. The computational cost of working with such a dense mesh is immeasurable and in practice one chooses a tolerance at which they are willing to deviate from the true solution so that the problem becomes tractable. This mitigation has plagued the FEM community since its inception and appeared to be an unavoidable trade off between computational complexity and accuracy.

In this paper we show that in fact error analysis of FEM has sunken into a tempting local minima; however, by resisting the gradient we can discover the globally minimal error solution to any PDE by going against conventional wisdom. We show that in the limit as the underlying mesh density is decreased, we also recover a zero error solution thus achieving a perfect solution to any PDE at more than reasonable computational cost. This technique is based heavily on the theoretical contributions of functional analysis of ***The Big Point*** in the seminal work [Yang 2021 or 2022].

DOI: http://dx.doi.org/10.1145/(to come)

2. Preliminaries

In this section we summarize some actual math. SIGTBD reviewers can just skip this section. A comprehensive tutorial of FEM is beyond the scope of this paper, and for interested readers we recommend the survey papers by [**?**] or classic textbooks [**? ?**]. We will now go on to provide an almost comprehensive tutorial of FEM.

One begins with a *strong form* PDE such as:

$$\Delta x = b$$

where x is the desired solution and b encodes given boundary conditions or external forcing terms. Next we integrate it against a set of test functions v from a yet unspecified space: resulting in the *weak form* equation.

$$\int_\Omega v\Delta x = \int_\Omega vb$$

Note that if v is chosen as the space of delta functions whose support is in Ω, the weak form is essentially equivalent to the strong form.

In practice, $\Omega \subset R^2$ is discretized by the union of many little non-intersecting polygons: a polygonal mesh. Often these polygon are triangles or quadrilaterals. In such a setting, v is chosen to be in the space of functions that can be parameterized by values on only a finite number of special points N on the mesh. This reduces the size of the test function space from infinite dimensional to finite dimensional. Now it is feasible to construct a linear basis of the function space

$$V = \{v \mid v = \sum_{i=1}^{N} a_i \phi_i\}.$$

At this point we ask that the solution x also be in the function space V: $x = \sum_j \phi_j b_j$. Our goal is simply to solve for the vector b that parameterizes solution x. Finally, the weak form equation can be expanded into a set of equations of the form

$$\forall i \int_\Omega \phi_i \Delta x = \int_\Omega \phi_i b$$

$$\forall i \int_\Omega \phi_i \Delta \sum_j \phi_j b_j = \int_\Omega \phi_i b$$

By performing mathematical operations on this weak form we can obtain the final result

$$Ab = c$$

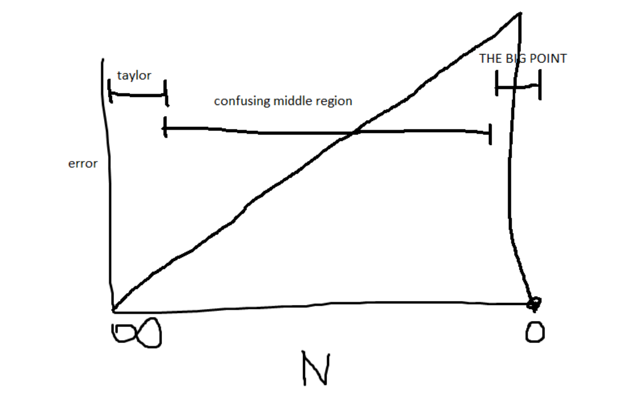

Figure 1. One can see that as N increases the FEM promise of decreasing error holds. However, we can also see that as N decreases to 1 (there's a typo in the figure but it's too expensive to re-render at this point), we achieve global optimality and obtain 0 error at much cheaper computational cost.

3. Theoretical Analysis

Note that the size of the system $Ab = c$ is $N \times N$ and that in order to obtain the strong form solution we can let $N \to \infty$. **This is where we deviate significantly from previous work.** Consider instead what happens instead when $N \to 1$. A zero dimensional function space V only consists of

$$V = \{0\}$$

and will be referred to as the ***The Big Point*** In this setting, the solution is immediately obvious to the casual observer $b = 0$. Therefore, $x = 0$ and the discrete forcing and boundary conditions c are also 0. In fact even the discretized laplacian operator A becomes just 0. This is an immediate result of discretizing one's solution, boundary, and error functions by the special function space ***The Big Point***.

In the context of the polygonal mesh of Ω, this corresponds to representing our domain with a single point. A corollary of this approach is that all existing point cloud processing techniques can now be applied trivially.

4. Empirical Error

We present results for the previously depicted PDE computed on a typical laptop in figure 1. Numerical error is depicted for a variety of values of N covering all three regimes of discretization: *taylor*, *middle*, *TBP*. In the first regime, the taylor series analysis of typical FEM error analysis kicks in resulting in a decreasing error function. In the region where taylor series analysis is inapplicable because of how sparse the mesh is, error analysis cannot be performed. Finally as N approaches 1 we enter ***The Big Point*** region where error is forced to 0 as well as all other quantities.

5. Adaptive Mesh De-resolution

This evidence suggests the development of a novel algorithm for obtaining 0 error solutions. We start the FEM method at an initial N and at each iteration, instead of solving the dense mesh equation $Ab = c$, we re-discretize by recursive formula $N = ceil(N/2)$. Once the initial mesh has converged to a single point, we have reached the domain of the big point. Following this procedure provably results in 0 error.

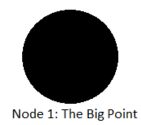

Figure 2. A polygonal mesh of the Mandelbrot fractal by a single point.

6. Conclusion and Discussion

In this paper we have discovered a global minima to the FEM problem that can be computed easily on even a potato with 0 stamped on it. We make use of ***The Big Point*** function space which greatly simplifies all the analytical details previous FEM method faced. Furthermore, we have uncovered a new frontier in the range of mesh discretization. While it was previously believed meshes could only be dense enough for taylor series approximations, or not, we have discovered a new regime in which a mesh can simply be ***The Big Point***. This work simultaneously addresses and concludes all numerical simulation fields of research as well as any ongoing meshing research. The simplicity and superiority of this method makes it widely and easily adoptable.

References

[Yang 2021 or 2022] J. Yang, P. Zhang, and P. robably'Erdos. The Big Point Theorem: Theory and Applications. *SIGTBD*, 1–9001, April 1.

7. Appendix A: The mesh

We depict the mesh used for the big point simulation in figure 2.